Key Concept

In Part 1 we discuss situations in which the standard deviations of the two populations are unknown and are not assumed to be equal.

In Part 2 we discuss two other situations: (1) The two population standard deviations are both known; (2) the two population standard deviations are unknown but are assumed to be equal.

Because \( \sigma \) is typically unknown in real situations, most attention should be given to the methods described in Part 1.

Part 1: Independent Samples with \( \sigma_1 \) and \( \sigma_2 \) Unknown and Not Assumed Equal
Definitions

Two samples are **independent** if the sample values selected from one population are not related to or somehow paired or matched with the sample values from the other population.

Two samples are **dependent** if the sample values are paired. (That is, each pair of sample values consists of two measurements from the same subject (such as before/after data), or each pair of sample values consists of matched pairs (such as husband/wife data), where the matching is based on some inherent relationship.)

Notation

\[
\begin{align*}
\mu_1 &= \text{population mean} \\
\sigma_1 &= \text{population standard deviation} \\
n_1 &= \text{size of the first sample} \\
\bar{x}_1 &= \text{sample mean} \\
s_1 &= \text{sample standard deviation} \\
\end{align*}
\]

Corresponding notations for \(\mu_2, \sigma_2, s_2, \bar{x}_2\) and \(n_2\) apply to population 2.

Requirements

1. \(\sigma_1\) and \(\sigma_2\) are unknown and no assumption is made about the equality of \(\sigma_1\) and \(\sigma_2\).
2. The two samples are independent.
3. Both samples are simple random samples.
4. Either or both of these conditions are satisfied: The two sample sizes are both large (with \(n_1 > 30\) and \(n_2 > 30\)) or both samples come from populations having normal distributions.

Hypothesis Test for Two Means: Independent Samples

\[
t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}
\]

(\(\mu_1 - \mu_2\) is often assumed to be 0)
Hypothesis Test - cont

Test Statistic for Two Means: Independent Samples

Degrees of freedom: In this book we use this simple and conservative estimate:
\[ df = \text{smaller of } n_1 - 1 \text{ and } n_2 - 1. \]

P-values: Refer to Table A-3. Use the procedure summarized in Figure 8-5.

Critical values: Refer to Table A-3.

Caution

Before conducting a hypothesis test, consider the context of the data, the source of the data, the sampling method, and explore the data with graphs and descriptive statistics. Be sure to verify that the requirements are satisfied.

Confidence Interval Estimate of \( \mu_1 - \mu_2 \): Independent Samples

\[
(\bar{x}_1 - \bar{x}_2) - E < (\mu_1 - \mu_2) < (\bar{x}_1 - \bar{x}_2) + E
\]

where \[ E = t_{\alpha/2} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \]

where df = smaller \( n_1 - 1 \) and \( n_2 - 1 \)

Example:

A headline in USA Today proclaimed that “Men, women are equal talkers.” That headline referred to a study of the numbers of words that samples of men and women spoke in a day. Given below are the results from the study. Use a 0.05 significance level to test the claim that men and women speak the same mean number of words in a day. Does there appear to be a difference?

<table>
<thead>
<tr>
<th>Number of Words Spoken in a Day</th>
<th>Men</th>
<th>Women</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n_1 )</td>
<td>186</td>
<td>210</td>
</tr>
<tr>
<td>( \bar{x}_1 )</td>
<td>15,668.5</td>
<td>16,215.0</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>8632.5</td>
<td>7301.2</td>
</tr>
</tbody>
</table>
Example:

Requirements are satisfied: two population standard deviations are not known and not assumed to be equal, independent samples, simple random samples, both samples are large.

Step 1: Express claim as \( \mu_1 = \mu_2 \).

Step 2: If original claim is false, then \( \mu_1 \neq \mu_2 \).

Step 3: Alternative hypothesis does not contain equality, null hypothesis does.

\[ H_0 : \mu_1 = \mu_2 \text{ (original claim) } \quad H_1 : \mu_1 \neq \mu_2 \]

Proceed assuming \( \mu_1 = \mu_2 \) or \( \mu_1 - \mu_2 = 0 \).

Example:

Step 4: Significance level is 0.05

Step 5: Use a t distribution

Step 6: Calculate the test statistic

\[
t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}
\]

\[
\frac{15,668.5 - 16,210.0}{\sqrt{\frac{8632.5^2}{186} + \frac{7301.2^2}{210}}} = -0.676
\]

Example:

Use Table A-3: area in two tails is 0.05, df = 185, which is not in the table, the closest value is \( t = \pm 1.972 \)

Example:

Step 7: Because the test statistic does not fall within the critical region, fail to reject the null hypothesis:

\( \mu_1 = \mu_2 \) (or \( \mu_1 - \mu_2 = 0 \)).

There is not sufficient evidence to warrant rejection of the claim that men and women speak the same mean number of words in a day. There does not appear to be a significant difference between the two means.
Example:

Using the sample data given in the previous Example, construct a 95% confidence interval estimate of the difference between the mean number of words spoken by men and the mean number of words spoken by women.

<table>
<thead>
<tr>
<th>Number of Words Spoken in a Day</th>
<th>Men</th>
<th>Women</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_1$</td>
<td>186</td>
<td>210</td>
</tr>
<tr>
<td>$\bar{x}_1$</td>
<td>15,668.5</td>
<td>$\bar{x}_2$ = 16,215.0</td>
</tr>
<tr>
<td>$s_1$</td>
<td>8632.5</td>
<td>$s_2$ = 7301.2</td>
</tr>
</tbody>
</table>

Example:

Requirements are satisfied as it is the same data as the previous example.

Find the margin of Error, $E$; use $t_{a/2} = 1.972$

$E = t_{a/2} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = 1.972 \sqrt{\frac{8632.5^2}{186} + \frac{7301.2^2}{210}} = 1595.4$

Construct the confidence interval use $E = 1595.4$ and $\bar{x}_1 = 15,668.5$ and $\bar{x}_2 = 16,215.0$.

$(\bar{x}_1 - \bar{x}_2) - E < (\mu_1 - \mu_2) < (\bar{x}_1 - \bar{x}_2) + E$

$-2141.9 < (\mu_1 - \mu_2) < 1048.9$

Example:

Step 4: Significance level is 0.05

Step 5: Use a t distribution

Step 6: Calculate the test statistic

$t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$

$t = \frac{(15,668.5 - 16,215.0)}{\sqrt{\frac{8632.5^2}{186} + \frac{7301.2^2}{210}}} = -0.676$

Example:

Use Table A-3: area in two tails is 0.05, df = 185, which is not in the table, the closest value is $t = \pm 1.972$

Example:
Step 7: Because the test statistic does not fall within the critical region, fail to reject the null hypothesis:
\[ \mu_1 = \mu_2 \quad \text{or} \quad (\mu_1 - \mu_2) = 0. \]

There is not sufficient evidence to warrant rejection of the claim that men and women speak the same mean number of words in a day. There does not appear to be a significant difference between the two means.

Part 2: Alternative Methods
Independent Samples with \( \sigma_1 \) and \( \sigma_2 \) Known.

Requirements
1. The two population standard deviations are both known.
2. The two samples are independent.
3. Both samples are simple random samples.
4. Either or both of these conditions are satisfied: The two sample sizes are both large (with \( n_1 > 30 \) and \( n_2 > 30 \)) or both samples come from populations having normal distributions.

Hypothesis Test for Two Means: Independent Samples with \( \sigma_1 \) and \( \sigma_2 \) Both Known

\[
Z = \frac{ \bar{x}_1 - \bar{x}_2 - (\mu_1 - \mu_2) }{ \sqrt{ \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} } }
\]

\( P \)-values and critical values: Refer to Table A-2.

Confidence Interval: Independent Samples with \( \sigma_1 \) and \( \sigma_2 \) Both Known

\[
(\bar{x}_1 - \bar{x}_2) - E < (\mu_1 - \mu_2) < (\bar{x}_1 - \bar{x}_2) + E
\]

where
\[
E = z_{\alpha/2} \sqrt{ \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} }
\]
Methods for Inferences About Two Independent Means

Figure 9-3

Assume that \( \sigma_1 = \sigma_2 \) and Pool the Sample Variances.

Requirements
1. The two population standard deviations are not known, but they are assumed to be equal. That is \( \sigma_1 = \sigma_2 \).
2. The two samples are independent.
3. Both samples are simple random samples.
4. Either or both of these conditions are satisfied: The two sample sizes are both large (with \( n_1 > 30 \) and \( n_2 > 30 \)) or both samples come from populations having normal distributions.

Hypothesis Test Statistic for Two Means: Independent Samples and \( \sigma_1 = \sigma_2 \)

\[
t = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}
\]

Where

\[
s^2 = \frac{(n_1 - 1) S_1^2 + (n_2 - 1) S_2^2}{n_1 + n_2 - 2}
\]

and the number of degrees of freedom is \( df = n_1 + n_2 - 2 \)

Confidence Interval Estimate of \( \mu_1 - \mu_2 \):
Independent Samples with \( \sigma_1 = \sigma_2 \)

\[
(\bar{x}_1 - \bar{x}_2) - E < (\mu_1 - \mu_2) < (\bar{x}_1 - \bar{x}_2) + E
\]

where

\[
E = t_{\alpha/2} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}
\]

and number of degrees of freedom is \( df = n_1 + n_2 - 2 \)
**Strategy**

Unless instructed otherwise, use the following strategy:
Assume that \( \sigma_1 \) and \( \sigma_2 \) are unknown, do **not** assume that \( \sigma_1 = \sigma_2 \), and use the test statistic and confidence interval given in Part 1 of this section. (See Figure 9-3.)

**Recap**

In this section we have discussed:
- Independent samples with the standard deviations unknown and not assumed equal.
- Alternative method where standard deviations are known
- Alternative method where standard deviations are assumed equal and sample variances are pooled.

---

**Independent and Dependent Samples.** In Exercises 5–8, determine whether the samples are independent or dependent. 

486/8. Home Sales Data Set 23 in Appendix B includes the list price and selling price for each of 40 randomly selected homes.

**Dependent**, since the selling price is typically the list price reduced by a relatively small amount - i.e., the selling price is typically based on the list price.

486/8. Voltage On each of 40 different days, the author measured the voltage supplied to his home and he also measured the voltage produced by his gasoline-powered generator. (The data are listed in Data Set 13 in Appendix B.) One sample consists of the voltages in his home and the second sample consists of the voltages produced by the generator.

**CORRECTION:** Thanks questioning my answer. The following is my corrected answer. **Independent**, since there is no relationship between the voltage supplied to the house by the power company and the voltage generated by a completely separate gasoline-powered generator.
In Exercises 9–32, assume that the two samples are independent simple random samples selected from normally distributed populations. Do not assume that the population standard deviations are equal, unless your instructor stipulates otherwise.

487/11. Confidence Interval for Cigarette Tar
The mean tar content of a simple random sample of 25 unfiltered king size cigarettes is 21.1 mg, with a standard deviation of 3.2 mg. The mean tar content of a simple random sample of 25 filtered 100 mm cigarettes is 13.2 mg with a standard deviation of 3.7 mg (based on data from Data Set 4 in Appendix B). Construct a 90% confidence interval estimate of the difference between the mean tar content of unfiltered king size cigarettes and the mean tar content of filtered 100 mm cigarettes. Does the result suggest that 100 mm filtered cigarettes have less tar than unfiltered king size cigarettes?

Yes; since the confidence interval includes only positive values, the results suggest that the filtered cigarettes have less tar than the unfiltered ones. That is, \( \mu_1 - \mu_2 > 0 \) which means \( \mu_1 > \mu_2 \).

487/12. Hypothesis Test for Cigarette Tar
Refer to the sample data in Exercise 11 and use a 0.05 significance level to test the claim that unfiltered king size cigarettes have a mean tar content greater than that of filtered 100 mm cigarettes. What does the result suggest about the effectiveness of cigarette filters?

In Exercises 9–32, assume that the two samples are independent simple random samples selected from normally distributed populations. Do not assume that the population standard deviations are equal, unless your instructor stipulates otherwise.

487/15. Hypothesis Test for Heights of Supermodels
The heights are measured for the simple random sample of supermodels Crawford, Bundchen, Pestova, Christenson, Hume, Moss, Campbell, Schiffer, and Taylor. They have a mean of 70.0 in. and a standard deviation of 1.5 in. Data Set 1 in Appendix B lists the heights of 40 women who are not supermodels, and they have heights with a mean of 63.2 in. and a standard deviation of 2.7 in. Use a 0.01 significance level to test the claim that the mean height of supermodels is greater than the mean height of women who are not supermodels.

Since the confidence interval includes only positive values, the results suggest that the mean height of supermodels is greater than the mean height of women who are not supermodels. That is, \( \mu_1 - \mu_2 > 0 \) which means \( \mu_1 > \mu_2 \).

487/16. Confidence Interval for Heights of Supermodels
Use the sample data from Exercise 15 to construct a 98% confidence interval for the difference between the mean height of supermodels and the mean height of women who are not supermodels. What does the result suggest about those two means?

Yes; since the confidence interval includes only positive values, the results suggest that the filtered cigarettes have less tar than the unfiltered ones. That is, \( \mu_1 - \mu_2 > 0 \) which means \( \mu_1 > \mu_2 \).
In Exercises 9–32, assume that the two samples are independent simple random samples selected from normally distributed populations. Do not assume that the population standard deviations are equal, unless your instructor stipulates otherwise.

**488/30a. Radiation in Baby Teeth**

Listed below are amounts of strontium-90 (in millibecquerels or mBq per gram of calcium) in a simple random sample of baby teeth obtained from Pennsylvania residents and New York residents born after 1979 (based on data from “An Unexpected Rise in Strontium-90 in U. S. Deciduous Teeth in the 1990s,” by Mangano, et al., Science of the Total Environment).

Use a 0.05 significance level to test the claim that the mean amount of strontium-90 from Pennsylvania residents is greater than the mean amount from New York residents.

Pennsylvania: 155 142 149 130 151 163 151 142 156 133 138 161
New York: 133 140 142 131 134 129 128 140 140 140 137 143

**488/30b. Radiation in Baby Teeth**

Listed below are amounts of strontium-90 (in millibecquerels or mBq per gram of calcium) in a simple random sample of baby teeth obtained from Pennsylvania residents and New York residents born after 1979 (based on data from “An Unexpected Rise in Strontium-90 in U. S. Deciduous Teeth in the 1990s,” by Mangano, et al., Science of the Total Environment).

Construct a 90% confidence interval of the difference between the mean amount of strontium-90 from Pennsylvania residents and the mean amount from New York residents.

Pennsylvania: 155 142 149 130 151 163 151 142 156 133 138 161
New York: 133 140 142 131 134 129 128 140 140 140 137 143

Since the confidence interval includes only positive values, $\mu_1 - \mu_2 > 0$ which means $\mu_1 > \mu_2$. That is Pennsylvania has a larger mean than New York’s mean amount of strontium-90.

**Pooling.** In Exercises 37–40, assume that the two samples are independent simple random samples selected from normally distributed populations. Also assume that the population standard deviations are equal ($\sigma_1 = \sigma_2$), so that the standard error of the differences between means is obtained by pooling the sample variances as described in Part 2 of this section.

**490/39. Confidence Interval with Pooling**

Repeat Exercise 11 with the additional assumption that $\sigma_1 = \sigma_2$. How are the results affected by this additional assumption?

Since the confidence interval includes only positive values, the results suggest that the filtered cigarettes have less tar than the unfiltered ones. That is, $\mu_1 - \mu_2 > 0$ which means $\mu_1 > \mu_2$. Problem 11 has confidence interval of $(6.2584, 9.5416)$ which is slightly wider than the pooled confidence interval. The final results are unchanged.

**Pooling.** In Exercises 37–40, assume that the two samples are independent simple random samples selected from normally distributed populations. Also assume that the population standard deviations are equal ($\sigma_1 = \sigma_2$), so that the standard error of the differences between means is obtained by pooling the sample variances as described in Part 2 of this section.

**490/40. Confidence Interval with Pooling**

Repeat Exercise 12 with the additional assumption that $\sigma_1 = \sigma_2$. How are the results affected by this additional assumption?

Since the P-value $\approx 0.0 < \alpha = 0.05$, reject $H_0: \mu_1 - \mu_2 = 0$. There is sufficient evidence to conclude that $\mu_1 > \mu_2$. There is sufficient evidence to support the claim that unfiltered king size cigarettes have a mean tar content greater than that of filtered 100 mm cigarettes. The results suggest that filters are effective in reducing the tar content in cigarettes.