4.6 Polynomial and Rational Inequalities

- Solve polynomial and rational inequalities.

Polynomial Inequalities

A quadratic inequality can be written in the form $ax^2 + bx + c > 0$, where the symbol $>$ could be replaced with either $<, \geq,$ or $\leq$.

A quadratic inequality is one type of polynomial inequality.

Examples of polynomial inequalities:

- $-2x^2 - x - 2 > 4$
- $\frac{3}{4}x + 6 \leq 0$
- $8x^3 + 2x^2 < 6x + 5$

Example

Solve: $x^2 - 3x - 4 > 0$.

Let’s look at the graph of $f(x) = x^2 - 3x - 4$.

The zeros are 4 and -1. Thus the x-intercepts of the graph are (-1,0) and (4,0).

I suggest that you watch these videos if you have any difficulty with this section.

- Polynomial inequalities - 5-minute video from youtube using algebraic method with test points.
- Example of Solving a Rational Inequality - 2-minute video from youtube using algebraic method and multiplicity. This is a REALLY good shortcut method.
- Solving rational inequality - 5-minute video from youtube using a combination of algebraic and calculator methods.
- Graphing rational functions (6 minutes) - http://www.youtube.com/watch?v=YyNNdBrv6VA
Example

The zeros divide the x-axis into intervals:

The sign of the function is the same for all values of x in a given interval. We can choose a test value for x from each interval and find the sign of f(x).

Test values: f(-2) = -6 \quad f(0) = -4 \quad f(5) = 6

The solution set consists of the intervals where the sign of f(x) is positive. So the solution set is \{x | x < -1 \text{ or } x > 4\}.

To Solve a Polynomial Inequality

1. Find an equivalent inequality with P(x) on one side and 0 on one side.
2. Solve the related polynomial equation; that is solve for f(x) = 0.
3. Use the solutions to divide the x-axis into intervals. Then select a test value from each interval and determine the polynomial’s sign on the interval.
4. Determine the intervals for which the inequality is satisfied and write interval notation or set-builder notation for the solution set. Include the endpoints of the intervals in the solution set if the inequality symbol is ≤ or ≥.

Example

Solve: \(4x^3 - 7x^2 - 15x \leq 0\).

We need to find all the zeros of the function so we solve the related equation.

\[4x^3 - 7x^2 - 15x = 0\]
\[x(4x^2 - 7x - 15) = 0\]
\[x(4x + 5)(x - 3) = 0\]

The zeros are -5/4, 0, and 3. Thus the x-intercepts of the graph are (-5/4, 0), (0, 0), and (3, 0).

Example continued

The zeros divide the x-axis into intervals.

For all x-values within a given interval, the sign of \(4x^3 - 7x^2 - 15x\) ≤ 0 must be either positive or negative. To determine which, we choose a test value for x from each interval and find f(x).

Interval \quad Test Value \quad Sign of f(x)
\[(-\infty, -5/4) \quad f(-2) = -30\quad \text{Negative}\]
\[(-5/4, 0) \quad f(-1) = -4\quad \text{Positive}\]
\[0, 3) \quad f(1) = -18\quad \text{Negative}\]
\[(3, \infty) \quad f(4) = 84\quad \text{Positive}\]

Since we are solving \(4x^3 - 7x^2 - 15x \leq 0\), the solution set consists of only two of the four intervals, those in which the sign of f(x) is negative \(\{x | -5/4 < x < -3 \text{ or } 0 < x < 3\}\).
To Solve a Rational Inequality

1. Find an equivalent inequality with 0 on one side.
2. Change the inequality symbol to an equals sign and solve the related equation, that is, solve \( f(x) = 0 \). Find \( x \) for numerator = 0.
3. Find the values of the variable for which the related rational function is not defined. Find \( x \) for denominator = 0.
4. The numbers found in steps (2) and (3) are called critical values. Use the critical values to divide the x-axis into intervals. Then test an \( x \)-value from each interval to determine the function’s sign in that interval.
5. Select the intervals for which the inequality is satisfied and write interval notation or set-builder notation for the solution set. If the inequality symbol is \( \leq \) or \( \geq \), then the solutions to step (2) should be included in the solution set. The \( x \)-values found in step (3) are never included in the solution set.

Example

Solve \( f(x) = \frac{x + 3}{x^2 - 1} \geq 0 \).

The denominator tells us that \( f(x) \) is not defined for \( x = 1 \) and \( x = -1 \).

Next, solve the related equation.

\[
\frac{x + 3}{x^2 - 1} = 0
\]

\[
\frac{x + 3}{x^2 - 1} = 0
\]

\[
x + 3 = 0
\]

\[
x = -3
\]

\[
\text{NOTE: Since the fraction equals zero, the numerator must equal zero.}
\]

\[
x + 3 = 0
\]

\[
x = -3
\]

The critical values are \(-3, -1, \) and \(1\). These values divide the x-axis into four intervals. We use a test value to determine the sign of \( f(x) \) in each interval.

Example continued

The critical values are \(-3, -1, \) and \(1\). These values divide the x-axis into four intervals. We use a test value to determine the sign of \( f(x) \) in each interval.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Test Value</th>
<th>Sign of ( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>((-\infty, -3))</td>
<td>(-4)</td>
<td>(-1/15)</td>
</tr>
<tr>
<td>((-3, -1))</td>
<td>(-2)</td>
<td>(1/3)</td>
</tr>
<tr>
<td>((-1, 1))</td>
<td>(0)</td>
<td>(-3)</td>
</tr>
<tr>
<td>((1, \infty))</td>
<td>(2)</td>
<td>(5/3)</td>
</tr>
</tbody>
</table>

Example continued

\( f(x) = \frac{x + 3}{x^2 - 1} \geq 0 \)

Function values are positive in the intervals \((-3, -1)\) and \((1, \infty)\). Since \(-1\) is not in the domain of \( f(x) \), it cannot be part of the solution set. Note that \(-3\) does satisfy the inequality.

The solution set is \([-3, -1) \cup (1, \infty)\).
Summary of procedure:

Algebraic Method
(1) Combine terms to get all terms (a simple fraction) on one side of the inequality with zero on the other size.
(2) Factor, set each factor equal to zero, and solve for x.
(3) Set the numerator equal to zero and solve for x.
(4) Set the denominator equal to zero and solve for x.
(5) Put each of the x-values from (2) and (3) above on the number line from smallest to largest from left to right.
(6) Determine what intervals of number values satisfy the original inequality.

Graphic Method
(1) Combine terms to get all terms (a simple fraction) on one side of the inequality with zero on the other size.
(2) Graph the function and find the zeros (roots) with the CALC function.
(3) Put each of the x-values from (2) above on the number line from smallest to largest from left to right.
(4) Determine what intervals of number values satisfy the original inequality.

368/2. For the function \( f(x) = x^2 + 2x - 15 \), solve \( f(x) < 0 \).

![Graph of f(x) = x^2 + 2x - 15 with intervals shown]

Function values are negative only in the interval (−5, 3). The solution interval is (−5, 3).

368/9. For the function \( g(x) = (x - 2) / (x + 4) \), solve \( g(x) \geq 0 \).

![Graph of g(x) = (x - 2) / (x + 4) with intervals shown]

Numerator = 0 gives x = 2; denominator = 0 gives x = -4.

The critical value -4 is not included because it is not in the domain of the function. The solution set is (−∞, -4) \( \cup \) [2, ∞).

368/14. For the function \( h(x) = 7x / (x - 1)(x + 5) \), solve \( h(x) > 0 \).

![Graph of h(x) = 7x / (x - 1)(x + 5) with intervals shown]

Numerator = 0 gives x = 0; denominator = 0 gives x = 1 and x = -5.

The critical values 1 and -5 are not included because they are not in the domain of the function. The solution set is (−5, 0) \( \cup \) (1, ∞).
369/18. For the function \( g(x) = x^5 - 9x^3 \), solve \( g(x) \leq 0 \).

The critical values -3, 0, and 3 are included because the inequality contains the possibility of equals. The solution set is \( (-\infty, -3] \cup [0, 3] \).

The factored form is \( x^3(x+3)(x-3) = 0 \). The factors = 0 yield \( x = 0 \), \( x = -3 \), and \( x = 3 \).

369/24. A related function is graphed. Solve the given inequality:
\[
\frac{8}{x^2 - 4} < 0.
\]

The factored form is \( x^2 - 4 = (x + 2)(x - 2) = 0 \) gives \( x = -2 \) and \( x = 2 \).

The critical values 2 and -2 are not included because they are not in the domain of the function. The solution set is \( (-2, 2) \).

The numerator = 0 is impossible; denominator = \( (x + 2)(x - 2) = 0 \) gives \( x = -2 \) and \( x = 2 \).

369/30. Solve \( x^2 + 4x + 7 \geq 5x + 9 \).

The factored form is \( (x+2)(x+1) = 0 \). The factors = 0 yield \( x = -2 \) and \( x = -1 \).

The critical values 2 and -1 are included because the inequality contains the possibility of equals.

The solution set is \( (-\infty, -1] \cup [2, \infty) \).

369/40. Solve \( 2x^3 - x^2 < 5x \).

The factored form is \( x^2(x - 5) = 0 \). The factors = 0 yield \( x = 0 \) and the approximate values \( x = -1.3508 \) and \( x = 1.8508 \), by calculator or quadratic formula.

The critical values 0, -1.3508, and 1.8508 are not included because the inequality does not contain the possibility of equals.

The solution set is approximately \( (-\infty, -1.3508) \cup (0, 1.8508) \).

This shades the graph where \( Y_1 < 0 \). Find the x-values for \( Y_1 = 0 \). Thus, the solution set is approximately \( (-\infty, -1.3508) \cup (0, 1.8508) \).
369/46. Solve \( x^5 + 24 > 3x^3 + 8x^2 \).

The factored form is \((x^2 - 8)(x^3 - 3) = 0\). The factors \( = 0\) yield \( x = 2, x = -\sqrt{3}, \) and \( x = \sqrt{3}\) or the approximate values \( x = -1.732 \) and \( x = 1.732\).

The critical values \( 2, -\sqrt{3}, \) and \( \sqrt{3}\) are not included because the inequality does not contain the possibility of equals. The solution set is \( (-\sqrt{3}, \sqrt{3}) \cup (2, \infty)\).

369/54. Solve \( \frac{1}{x - 3} \leq 3 \).

370/60. Solve \( \frac{x + 1}{x - 2} + \frac{x - 3}{x - 1} < 0 \).

The solution set is \( (1, 2) \).
370/68. Solve \( \frac{3}{x^2 - 4} \leq \frac{5}{x^2 + 7x + 10} \).

\[
\frac{3}{x^2 - 4} = \frac{5}{x^2 + 7x + 10} \\
\frac{3(x^2 + 7x + 10)}{x^2 - 4} = 5 \\
\frac{3x^2 + 21x + 30}{x^2 - 4} = 5 \\
x^2 + 7x + 10 = 5x^2 - 20 \\
4x^2 - 7x - 30 = 0 \\
(x - 3)(4x + 10) = 0 \\
x = 3, -\frac{5}{2} \\
\text{Solution set: } (-\infty, -\frac{5}{2}) \cup (-\frac{5}{2}, 3) \cup (3, \infty)
\]

The solution set is \((-\infty, -\frac{5}{2}) \cup (-\frac{5}{2}, 3) \cup (3, \infty)\).

370/72. Solve \( \frac{3}{x^2 + 3} > \frac{3}{5 + 4x^2} \).

\[
\frac{3}{x^2 + 3} > \frac{3}{5 + 4x^2} \\
3(5 + 4x^2) > 3(x^2 + 3) \\
15 + 12x^2 > 3x^2 + 9 \\
9x^2 > -6 \\
x^2 > -\frac{2}{3} \\
(\infty, \infty) \text{ solution} \\
\]

The solution set is \((\infty, \infty)\).

370/78. Population Growth. The population \( P \), in thousands, of Lordsburg is given by

\[
P(t) = \frac{500t}{(2t^2 + 9)} ,
\]

where \( t \) is the time, in months. Find the interval on which the population was 40,000 or greater. (See Exercise 81 in Exercise Set 4.5.)

\[
P(t) = 40,000 \\
\frac{500t}{2t^2 + 9} = 40 \\
500t = 800t^2 + 360 \\
800t^2 - 500t + 360 = 0 \\
(16t - 9)(50t - 40) = 0 \\
t = \frac{9}{16}, \frac{4}{5} \\
\text{Solution set: } [\frac{9}{16}, \frac{4}{5}] \\
\]

The solution set is \([\frac{9}{16}, \frac{4}{5}]\).

370/82. Number of Handshakes. If there are \( n \) people in a room, the number of possible handshakes by all the people in the room is given by the function

\[
N(n) = \frac{n(n - 1)}{2}.
\]

For what number \( n \) of people is \( 66 \leq N \leq 300 \)?

\[
66 \leq \frac{n(n - 1)}{2} \leq 300 \\
132 \leq n(n - 1) \leq 600 \\
132 \leq n^2 - n - 600 \leq 0 \\
(n - 13.5)^2 - 600 = 0 \\
(n - 13.5)^2 = 600 \\
(n - 13.5) = \pm 24.49 \\
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The solution set is \([-12.99, 50.49]\).

Below is graph above y = 0 (x-axis). The solution set is \([0.83, 5.42]\).

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